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Elementary operations and Laplace's theorem on quantum matrices

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Abstract. q-fundamental matrices are introduced and studied. Elementary operations on quantum matrices are discussed. The q-generalization of the classical Laplace's theorem is found. An application of the result is given.

1. Introduction and preparations

Recently, in growing interest in studying quantum groups from the physical and mathematical point of view, much research has been carried out on quantum matrices.

In this paper we introduce q-fundamental matrices, and study q-analogous elementary operations and the q-deformed Laplace's theorem on quantum matrices.

In general [1], for symbols U_i^α where $i \in \{1, \ldots, n\}$, $\alpha \in \{1, \ldots, m\}$ in the polynomial $\mathbb{C}$-algebra $A_q(m, n) = \mathbb{C}\langle U_i^\alpha\colon 1 \leqslant \alpha \leqslant m, 1 \leqslant i \leqslant n\rangle$ if the following commutative relations are satisfied:

$$U_i^\alpha U_i^\beta = q U_i^\beta U_i^\alpha \qquad \text{for } \alpha < \beta \tag{1a}$$

$$U_i^\alpha U_j^\alpha = q U_j^\alpha U_i^\alpha \qquad \text{for } i < j \tag{1b}$$

$$U_i^\alpha U_j^\beta = U_j^\beta U_i^\alpha \qquad \text{for } \alpha < \beta \text{ and } i > j \tag{1c}$$

$$U_i^\alpha U_j^\beta - U_j^\beta U_i^\alpha = (q - q^{-1}) U_j^\alpha U_i^\beta \qquad \text{for } \alpha < \beta \qquad i < j \tag{1d}$$

then the set of symbols $U = \{U_i^\alpha\}$ is called a quantum matrix of size $m \times n$. The set of all $m \times n$ quantum matrices is denoted by $M_q(m, n)$. Here the superscript α is the row index, the subscript i is the column index.

In Manin's approach [2, 3], the space $M_q(m, n)$ of quantum matrices of size $m \times n$ is shown as the space of algebra morphism from quantum planes $A_q^{m|o}$ and $A_q^{o|m}$ to, respectively, quantum planes $A_q^{n|o}$ and $A_q^{o|n}$ [1]. Here the quantum planes $A_q^{n|o}$ and $A_q^{o|n}$ are defined, respectively, as the polynomial $\mathbb{C}$-algebra $A_q^{n|o} = \mathbb{C}\langle x_i\colon i = 1, \ldots, n\rangle$ and $A_q^{o|n} = \mathbb{C}\langle y_i\colon i = 1, \ldots, n\rangle$ generated correspondingly by symbols x_i and y_i with the following commutation rules:

$$x_i x_j = q x_j x_i \qquad \text{for } i < j$$

$$y_i^2 = 0 \qquad y_i y_j = -q^{-1} y_j y_i \qquad \text{for } i < j. \tag{2}$$

When $m = n$, $M_q(n, n)$ and its coordinate ring $A_q(n, n)$ are denoted $M_q(n)$ and $A_q(n)$ for brevity. Thus, if $\{Z_i^k\} \in M_q(n)$, then the following hold:

$$Z_i^k Z_i^h = q Z_i^h Z_i^k \qquad \text{for } k < h \tag{3a}$$

$$Z_i^k Z_j^k = q Z_j^k Z_i^k \qquad \text{for } i \leqslant j \tag{3b}$$

$$Z_i^k Z_j^h - Z_j^h Z_i^k = (q - q^{-1}) Z_i^h Z_j^k \qquad \text{for } i<j \qquad k<h \tag{3c}$$

$$Z_i^k Z_j^h = Z_j^h Z_i^k \qquad \text{for } i<j \qquad k>h. \tag{3d}$$

The quantum determinant of the matrix $\{Z_i^k\}$ generating $A_q(n)$ is defined [4] by

$$\det_q\{Z_i^k\} = \sum_{\sigma\in S_n} (-q)^{t(\sigma)} Z_1^{\sigma(1)} Z_2^{\sigma(2)} \ldots Z_n^{\sigma(n)} \tag{4}$$

$$= \sum_{\sigma\in S_n} (-q)^{t(\sigma)} Z_{\sigma(1)}^1 Z_{\sigma(2)}^2 \ldots Z_{\sigma(n)}^n \tag{5}$$

where S_n is the permutation group of the set $\{1, 2, \ldots, n\}$ and for each $\sigma \in S_n$, $t(\sigma)$ denotes the number of inversions in the permutation $(\sigma(1) \ldots \sigma(n))$, i.e. the number of pairs (i, j) with $1 \leqslant i < j \leqslant n$ and $\sigma(i) > \sigma(j)$. Equation (3*d*) may be used to prove that (4) = (5).

To meet the needs below, we generalize the concept of quantum matrix as follows.

Definition 1. The set of symbols $X = \{X_i^\alpha\}$ ($\alpha \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$) in the polynomial $\mathbb{C}$-algebra $A(m, n) = \mathbb{C}\langle X_i^\alpha : 1 \leqslant \alpha \leqslant m,\ 1 \leqslant i \leqslant n\rangle$ is called a *q-fundamental matrix*. If $m = n$, then its determinant is defined by

$$\det_q\{X_i^j\} = \sum_{\sigma\in S_n} (-q)^{t(\sigma)} X_{\sigma(1)}^1 \ldots X_{\sigma(n)}^n$$

which is called a *q-fundamental determinant*.

In general, in definition 1

$$\det_q\{X_i^j\} \neq \sum_{\sigma\in S_n} (-q)^{t(\sigma)} X_1^{\sigma(1)} \ldots X_n^{\sigma(n)}.$$

It is easy to see that if the elements of a q-fundamental matrix satisfy the relations (1), then the matrix is a quantum one; moreover, if this is a square one, then its q-fundamental determinant is just its quantum determinant and is also equal to

$$\sum_{\sigma\in S_n} (-q)^{t(\sigma)} X_1^{\sigma(1)} \ldots X_n^{\sigma(n)}.$$

Definition 2. In a q-fundamental matrix $X = \{X_i^\alpha\}$ where $\alpha \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$, if $M_0 \subseteq \{1, \ldots, m\}$, $N_0 \subseteq \{1, \ldots, n\}$, then $X(M_0, N_0) = \{X_j^\beta\}$ ($\beta \in M_0, j \in N_0$) is called a *submatrix* of X. If $|M_0| = |N_0| = p$, then the q-fundamental determinant of $X(M_0, N_0)$ is called a *q-minor of the pth order of X*.

Obviously, if X is a quantum matrix, then any submatrix of X is also a quantum one; and in this situation, any q-minor of X is a quantum determinant.

It is not difficult to prove that if $U = \{U_i^\alpha\} \in M_q(m, n)$, then its transpose $U^{\mathrm{T}} = \{v_\alpha^i\} \in M_q(n, m)$ where $V_\alpha^i = U_i^\alpha$ for $1 \leqslant \alpha \leqslant m$, $1 \leqslant i \leqslant n$; and if $m = n$, then $\det_q U^{\mathrm{T}} = \det_q U$ because (4) = (5) on quantum matrices.

2. Elementary operations

The following three operations on q-fundamental matrices (in particular, on quantum matrices) are called the *elementary row* (*or column*) *operations* of types 1, 2 and 3 respectively:

(i) interchanging two rows (or columns);
(ii) multiplying all elements of a row (or column) by some non-zero number in $\mathbb{C}$;
(iii) adding to any row (or column) any other row (or column) multiplied by a non-zero number in $\mathbb{C}$.

Now observe that these manipulations of the rows (or columns) of a q-fundamental matrix can be achieved by premultiplication (or postmultiplication) of X by appropriate matrices. In particular, the interchange of rows i_1 and i_2 of X can be performed by multiplying X from the left by the $m \times m$ matrix

$$C(i_1, i_2) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 \cdots 1 & & & \\ & & & \vdots \quad \vdots & & & \\ & & & 1 \cdots 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}_{m \times m}$$

obtained by interchanging rows i_1 and i_2 of the identity matrix.

Furthermore, the effect of multiplying the ith row of X by a non-zero number k in $\mathbb{C}$ can be achieved by forming the product $M_i(k)X$ where

$$M_i(k) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & k & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} i.$$

Finally, adding $k(\in \mathbb{C})$ times row i_2 to row i_1 of X is equivalent to multiplication of X from the left by the matrix

$$A(k; i_2, i_1) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 \cdots k & & \\ & & \ddots \;\; \vdots & & \\ & & \;\; 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{matrix} i_1 \\ i_2 \end{matrix} \qquad (\text{if } i_1 < i_2)$$

$$\text{or} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 \ddots & \\ & & \vdots \quad \ddots & \\ & & k \cdots 1 & \\ & & & 1 \end{pmatrix} \begin{matrix} i_2 \\ i_1 \end{matrix} \qquad (\text{if } i_2 < i_1).$$

Similarly, the multiplication of X on the right by the appropriate matrices $C(i_1, i_2)$, $M_i(k)$ or $A(k; i_2, i_1)$ leads to analogous changes in columns.

The matrices $C(i_1, i_2)$, $M_i(k)$ and $A(k; i_2, i_1)$ are called *elementary matrices* of types 1, 2 and 3, respectively.

Note that in general, those matrices achieved by three elementary operations on quantum matrices are not quantum matrices, are only q-fundamental matrices. Now, we discuss the determinents of such some q-fundamental matrices.

For a fixed quantum matrix $\{Z_j^i\} \in M_q(n)$, there exist algebra morphisms [1, 4]:

$$\delta_n: A_q^{n|o} \to A_q(n) \otimes A_q^{n|o} \qquad \bar{\delta}_n: A_q^{o|n} \to A_q(n) \otimes A_q^{o|n}.$$

The co-action $\bar{\delta}_n$ applied to the monomial $y_1, \ldots, y_n$ gives the formula [2, 4]

$$\bar{\delta}_n(y_1 \ldots y_n) = \det_q\{Z_j^i\} \otimes y_1 \ldots y_n \tag{6}$$

such that

$$\delta_n(x_i) = \sum_{j=1}^{n} Z_j^i \otimes x_j \qquad \bar{\delta}_n(y_i) = \sum_{j=1}^{n} Z_j^i \otimes y_i.$$

Analogously, applying $\bar{\delta}_n$ to the monomial $y_i \ldots y_{i_n}$ one finds

$$\bar{\delta}_n(y_{i_1} \ldots y_{i_n}) = \sum_{\sigma \in S_n} (-q)^{t(\sigma)} Z^{i_1}{}_{\sigma}(1) \ldots Z^{i_n}{}_{\sigma}(n) \otimes y_1 \ldots y_n. \tag{7}$$

If $y_{i_1} \ldots y_{i_n} = 0$, then $\bar{\delta}_n(y_{i_1} \ldots y_{i_n}) = 0$ and the following [1] holds:

$$\sum_{\sigma \in S_n} (-q)^{t(\sigma)} Z_{\sigma}^{i_1}(1) \ldots Z_{\sigma(n)}^{i_n} = 0$$

This means that, for a quantum matrix $\{Z_i^j\} \in M_q(n)$, the q-fundamental determinant of the q-fundamental matrix

$$\begin{pmatrix} Z_1^{i_1} \ldots Z_n^{i_n} \\ \ldots\ldots \\ Z_1^{i_n} \ldots, Z_n^{i_n} \end{pmatrix}$$

is equal to zero if $y_{i_1} \ldots y_{i_n} = 0$.

Property 1[1]. (i) For a set $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, n\}$, if two or more indices $i_1, \ldots, i_n$ coincide, then the determinant of

$$\begin{pmatrix} Z_1^{i_1} \ldots Z_n^{i_1} \\ \ldots\ldots \\ Z_1^{i_n} \ldots, Z_n^{i_n} \end{pmatrix}$$

vanishes, i.e.

$$\sum_{\sigma \in S_n} (-q)^{t(\sigma)} Z_{\sigma(1)}^{i_1} \ldots Z_{\sigma(n)}^{i_n} = 0.$$

if $i_k = i_h$ for some $k, h \in \{1, \ldots, n\}$.

(ii) If all indices $i_1, \ldots, i_n$ are distinct, then

$$\sum_{\sigma \in S_n} (-q)^{t(\sigma)} Z_{\sigma(1)}^{i_1} \ldots Z_{\sigma(n)}^{i_n} = (-q)^{t(i_1 \cdots i_n)} \det_q\{Z_i^j\}.$$

Proof. (i) According to (2), $y_{i_1} \ldots y_{i_n} = 0$.

(ii) By (2), $y_{i_1}\dots y_{i_n}=(-q)^{t(i_1\dots i_n)}y_1\dots y_n$, then $\tilde{\delta}_n(y_{i_1}\dots y_{i_n})=(-q)^{t(i_1\dots i_n)}\cdot\tilde{\delta}_n(y_1\dots y_n)$, and by (6) and (7), the equality follows.

This property means that the determinant of the q-fundamental matrix reduced by the application of an elementary row operation of type 1 to a quantum matrix is equal to the multipication of the quantum determinant by some power of $-q$.

The following is obvious:

Property 2. For any q-fundamental matrix

$$X=\{X_i^j\}_{i,j=1}^n$$

and any $k\in\mathbb{C}$, $i\in\{1,\dots,n\}$, $\det_q M_i(k)X=k\det_q X$.

Also, obviously, for q-fundamental matrices

$$X=\{X_i^j\}_{i,j=1}^n$$

and

$$\tilde{X}=\begin{pmatrix} X_1^1\dots\dots X_n^1\\ \dots\dots\\ \tilde{X}_1^i\dots\dots\tilde{X}_n^i\\ \dots\dots\\ X_1^n\dots\dots X_n^n\end{pmatrix}$$

let

$$Y=\begin{pmatrix} X_1^1\dots\dots X_n^1\\ \dots\dots\\ X_1^i+\tilde{X}_1^i\dots X_n^i+\tilde{X}_n^i\\ \dots\dots\\ X_1^n\dots\dots X_n^n\end{pmatrix}$$

then $\det_q Y=\det_q X+\det_q\tilde{X}$.

From this equality and by property 1 and 2, we have the following:

Property 3. For any quantum matrix $Z=\{Z_i^j\}\in M_q(n)$ and $k\in\mathbb{C}$, $i_1, i_2\in\{1,\dots,n\}$, $\det_q A(k;i_2,i_1)Z=\det_q Z$.

Property 1 and 3 do not hold for general q-fundamental matrices.

Proposition 4. For any $Z=\{Z_i^j\}\in M_q(n)$ and any $s,t\in\mathbb{C}$, $h,j\in\{1,\dots,n\}$ $(h<j)$, let a q-fundamental matrix be

$$X=\begin{pmatrix} Z_1^1\dots\dots\dots Z_n^1\\ \dots\dots\dots\\ Z_1^{i-1}\dots\dots\dots Z_n^{i-1}\\ Z_1^h-sZ_1^j\dots\dots\dots Z_n^h-sZ_n^j\\ Z_1^h-tZ_1^j\dots\dots\dots Z_n^h-tZ_n^j\\ Z_1^{i+2}\dots\dots\dots Z_n^{i+2}\\ \dots\dots\dots\\ Z_1^n\dots\dots\dots Z_n^n\end{pmatrix}$$

then when $sq=t$, $\det_q X=0$.

Proof. $\bar{\delta}_n(y_1 \dots y_{i-1}(y_h - sy_j)(y_h - ty_j)y_{i+2} \dots y_n) = \det_q X \otimes y_1 \dots y_n$, the left-hand side vanishes, because $(y_h - sy_j)(y_h - ty_j) = y_h^2 - sy_j y_h - ty_h y_j + sty_j^2 = sqy_h y_j - ty_h y_j = (sq - t)y_h y_j = 0$.

3. q-Deformed Laplace's theorem

For a quantum matrix $Z = \{Z_i^j\} \in M_q(n)$ and $p \in \{1, \dots, n\}$, $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$, $i_1 < \dots < i_p$, $\{j_1, \dots, j_p\} \subseteq \{1, \dots, n\}$, $j_1 < \dots < j_p$, denote the q-minor of the pth order of Z with row indices $i_1, \dots, i_p$ and column indices $j_1, \dots, j_p$ by

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}.$$

We also need the notion of *complementary* q-minor, that is, the determinant of the submatrix of the quantum matrix $Z \in M_q(n)$ resulting from the deletion of the rows and columns listed in

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}$$

Denote it by

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c.$$

Lemma 5. For a quantum matrix $Z = \{Z_i^j\} \in M_q(n)$ and $p \in \{1, \dots, n\}$

$$\det_q Z = \sum_{1 \le j_1 < \dots < j_n \le n} (-q)^s M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix}^c \tag{8}$$

where $s = j_1 + \dots + j_p - 1 - \dots - p$.

Proof. Firstly, fix the chosen $j_1, \dots, j_p$, let $\{i_1, \dots, i_{n-p}\} = \{1, \dots, n\} \setminus \{j_1, \dots, j_p\}$ such that $i_1 < \dots < i_{n-p}$. Then

$$M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix}^c$$

$$= \left(\sum_{\sigma_0} (-1)^{t(\sigma_0)} Z^1_{\sigma_0(j_1)} \dots Z^p_{\sigma_0(j_p)} \right) \left(\sum_{\sigma_1} (-1)^{t(\sigma_1)} Z^{p+1}_{\sigma_1(i_1)} \dots Z^n_{\sigma_1(i_{n-r})} \right)$$

$$= \sum_{\sigma_0, \sigma_1} (-1)^{t(\sigma_0) + t(\sigma_1)} Z^1_{\sigma_0(j_1)} \dots Z^p_{\sigma_0(j_p)} Z^{p+1}_{\sigma_1(i_1)} \dots Z^n_{\sigma_1(i_{n-p})}$$

where σ_0, σ_1 are any permutations respectively of $\{j_1, \dots, j_p\}$ and $\{i_1, \dots, i_{n-p}\}$. Hence

$$M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} 1 \dots p \\ j_1 \dots j_p \end{pmatrix}^c$$

consists of $p!(n-p)!$ distinct terms of $\det_q Z$ up to coefficients.

Moreover, the right-hand side of (8) consists of

$$\binom{n}{p} p!(n-p)! = n!$$

distinct terms of $\det_q Z$, i.e., the two sides of (8) consist of the same terms up to coefficients.

The coefficient of any term $Z^1_{\sigma_0(j_1)} \ldots Z^p_{\sigma_0(j_p)} Z^{p+1}_{\sigma_1(i_1)} \ldots Z^n_{\sigma_1(i_{n-p})}$ in the right-hand side of (8) is $(-q)^d$, where $d = t(\sigma_0) + t(\sigma_1) + j_1 + \cdots + j_p - 1 - \cdots - p$. In the left-hand side of (8), the coefficient of this term is $(-q)^f$, where

$$f = t(\sigma_0(j_1) \ldots \sigma_0(j_p)\sigma_1(i_1) \ldots \sigma_1(i_{n-p})) = t(\sigma_0) + t(\sigma_1) + r$$

$r \overset{\text{def}}{=}$ is the number of pairs $(\sigma_0(j_u), \sigma_1(i_v))$ with $1 \leqslant u \leqslant p$, $1 \leqslant v \leqslant n-p$ and $\sigma_0(j_u) > \sigma_1(i_v)$. But σ_0, σ_1 are permutations, respectively, of $\{j_1, \ldots, j_p\}$ and $\{i_1, \ldots, i_{n-p}\}$, so $\{\sigma_0(j_1), \ldots, \sigma_0(j_p)\} = \{j_1, \ldots, j_p\}$ and $\{\sigma_1(i_1), \ldots, \sigma_1(i_{n-p})\} = \{i_1, \ldots, i_{n-p}\}$, which means $r =$ the number of pairs (j_u, i_v) with $1 \leqslant u \leqslant p$, $1 \leqslant v \leqslant n-p$ and $j_u > i_v$.

Since $j_1 < \cdots < j_p$, for any $u \in \{1, \ldots, p\}$, the number of pairs (j_u, i_v) with $1 \leqslant v \leqslant n-p$ and $j_u > i_v$ is $j_u - u$. Hence

$$r = \sum_{u=1}^{p} (j_u - u).$$

It follows that $d = f$, i.e. in the two sides of (8), the coefficients of any term $Z^1_{\sigma_0(j_1)} \ldots Z^p_{\sigma_0(j_p)} Z^{p+1}_{\sigma_1(i_1)} \ldots Z^n_{\sigma_1(i_{n-p})}$ are equal to each other. Therefore, (8) holds.

Now, we consider how to construct the expansion formula of $\det_q Z$ when

$$M\begin{pmatrix} 1 \ldots p \\ j_1 \ldots j_p \end{pmatrix}$$

is replaced with

$$M\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}$$

where $\{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$ and $i_1 < \cdots < i_p$. In fact, we have the following:

Theorem 6 (Laplace's theorem). For a quantum matrix $Z = \{Z^j_i\} \in M_q(n)$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$, $i_1 < \cdots < i_p$

$$\det_q Z = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (-q)^s M\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix} M\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c$$

where $s = j_1 + \cdots + j_p - i_1 - \cdots - i_p$.

Proof. Let $\{i_{p+1}, \dots, i_n\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_p\}$ and $i_{p+1} < \cdots < i_n$. We consider the q-fundamental matrix

$$\bar{Z} = \begin{pmatrix} Z_1^{i_1}, \dots, Z_n^{i_1} \\ \dots\dots \\ Z_1^{i_n} \dots Z_n^{i_n} \end{pmatrix}.$$

Obviously, for any $1 \leqslant j_1 < \cdots < j_p \leqslant n$, Z and $\bar{Z}$ have both the q-minor

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c.$$

and its complementary q-minor in Z is equal to that in $\bar{Z}$, i.e.

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

By Lemma 5

$$\det_q \bar{Z} = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (-q)^d M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

where $d = j_1 + \cdots + j_p - 1 - \cdots - p$.

By property 1, $\det_q \bar{Z} = (-q)^{t(i_1 \dots i_n)} \det_q Z$. $t(i_1 \dots i_n) = t(i_1 \dots i_p) + t(i_{p+1} \dots i_n) + r$, where $r =$ the number of pairs (i_u, i_v) with $1 \leqslant u \leqslant p$, $p+1 \leqslant v \leqslant n$ and $i_u > i_v = i_1 + \cdots + i_p - 1 - \cdots - p$ as shown in the proof of lemma 5. But $t(i_1 \dots i_p) = t(i_{p+1} \dots i_n) = 0$. Hence $t(i_1 \dots i_n) = i_1 + \cdots + i_p - 1 - \cdots - p$. Thus

$$(-q)^r \det_q Z = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (-q)^d M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

and we obtain

$$\det_q Z = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (-q)^s M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c.$$

This theorem is the q-generalization of the classical Laplace's theorem [5], i.e. when $q = 1$, we obtain the classical Laplace's theorem.

According to this theorem, one can define the *complementary q-cofactor* to the q-minor

$$M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}$$

by

$$M^c\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} \overset{\text{def}}{=} (-q)^s M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

where $s=j_1+\cdots+j_p-i_1-\cdots-i_p$. Making use of this notation, we can represent theorem 6 as

$$\det_q Z=\sum_{1\leq j_1<\cdots<j_p\leq n} M\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix} M^c\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}.$$

In particular, for any $i\in\{1,\ldots,n\}$

$$M\begin{pmatrix} i \\ j \end{pmatrix}=Z^i_j.$$

Denote

$$M\begin{pmatrix} i \\ j \end{pmatrix}^c=M_{ij}$$

then

$$M^c\begin{pmatrix} i \\ j \end{pmatrix}=(-q)^{j-i}M_{ij}.$$

Let $A_{ij}=(-q)^{j-i}M_{ij}$, we then obtain

Corollary 7 (Cofactor expansion). For a quantum matrix $Z=\{Z^j_i\}\in M_q(n)$ and any $i\in\{1,\ldots,n\}$, $\det_q Z=Z^i_1A_{i1}+\cdots+Z^i_nA_{in}$.

Because of the duality of row and column in the definitions of quantum matrix and quantum determinant, one can get the dual Laplace's theorem and the dual cofactor expansion by replacing row and column with each other.

4. An application

Let $V=\mathbb{C}^n$, a matrix R of the form

$$R=\sum_{\substack{i\neq j\\ i,j=1}}^{n} e_{ii}\otimes e_{jj}+q\sum_{i=1}^{n} e_{ii}\otimes e_{ii}+(q-q^{-1})\sum_{1\leq j<i\leq n} e_{ij}\otimes e_{ji}$$

where $e_{ij}\in \mathrm{Mat}(C^n)$ are matrix units and $q\in C$, satisfies the Yang–Baxter equation, i.e. $R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$ (see [4]).

Let $A=A(R)$ be the associative algebra over $\mathbb{C}$ with the generators 1, t_{ij}, $i,j=1,\ldots,n$, satisfying the relations

$$RT_1T_2=T_2T_1R$$

where $T_1=T\otimes I$, $T_2=I\otimes T\in\mathrm{Mat}(V^{\otimes 2},A)$, $T=(t_{ij})^n_{i,j=1}\in M_q(n)$ and I is a unit matrix in $\mathrm{Mat}(V,\mathbb{C})$. Then [1], $A(R)$ is the algebra of functions on the q-deformation of the group $\mathrm{GL}(n,\mathbb{C})$ and denote it by $\mathrm{Fun}_q(\mathrm{GL}(n,\mathbb{C}))$, i.e. $A(R)=\mathrm{Fun}_q(\mathrm{GL}(n,\mathbb{C}))$. It can be verified that T is a quantum matrix.

By [4, Theorem 3], $\det_q T$ generates the centre of the algebra $\mathrm{Fun}_q(\mathrm{GL}(n,\mathbb{C}))$, i.e. $\mathrm{Cen}(A)=\mathrm{Cen}(\mathrm{Fun}_q(\mathrm{GL}(n,\mathbb{C})))=\mathbb{C}\det_q T$.

For the quantum matrix $T=(t_{ij})\in M_q(n)$ and p, $i_1,\ldots,i_p$, $j_1,\ldots,j_p$ having the same meaning as in section 3, denote the submatrix of the pth order of T with row indices $i_1,\ldots,i_p$ and column indices $j_1,\ldots,j_p$ by

$$T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}$$

and denote the submatrix of T resulting from the deletion of the rows and columns listed in

$$T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}$$

by

$$T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c.$$

By the remark in section 1

$$T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}$$

and

$$T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c$$

are both quantum matrices. The R matrix $R'=$

$$\sum_{\substack{i\neq j\\ i,j=1}}^{p} e_{ii}\otimes e_{jj}+q\sum_{i=1}^{p} e_{ii}\otimes e_{ii}+(q-q^{-1})\sum_{1\leq j<i\leq p} e_{ij}\otimes e_{ji}$$

is a Yang–Baxter operator, i.e. $R'_{12}R'_{13}R'_{23}=R'_{23}R'_{13}R'_{12}$ holds, and

$$R'\left(T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}\otimes I\right)\left(I\otimes T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}\right)=\left(I\otimes T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}\right)\left(T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}\otimes I\right)R'$$

and

$$R'\left(T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c\otimes I\right)\left(I\otimes T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c\right)$$
$$=\left(I\otimes T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c\right)\left(T\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c\otimes I\right)R'.$$

Let

$$A\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}$$

and

$$A\begin{pmatrix} i_1 \ldots i_p \\ j_1 \ldots j_p \end{pmatrix}^c$$

respectively, be the associative algebras over $\mathbb{C}$ with the generators 1 and the elements of

$$T\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}$$

and the elements of

$$T\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

respectively. Then by [4, theorem 3],

$$\mathrm{Cen}\left(A\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}\right) = \mathbb{C}\,\det_q T\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} = \mathbb{C}M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}$$

$$\mathrm{Cen}\left(A\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c\right) = \mathbb{C}\,\det_q T\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c = \mathbb{C}M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c.$$

By theorem 6

$$\det_q T = \sum_{1 \leqslant j_1 < \dots < j_p \leqslant n} (-q)^s M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix} M\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c$$

where $s = j_1 + \dots + j_p - i_1 - \dots - i_p$. Thus, we have the following relation:

$$\mathrm{Cen}(\mathrm{Fun}_q(\mathrm{GL}(n, C))) = \mathrm{Cen}(A)$$

$$= \sum_{1 \leqslant j_1 < \dots < j_p \leqslant n} (-q)^s\, \mathrm{Cen}\left(A\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}\right) \mathrm{Cen}\left(A\begin{pmatrix} i_1 \dots i_p \\ j_1 \dots j_p \end{pmatrix}^c\right).$$

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